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Cnoidal-type surface waves in deep water

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Two-dimensional potential flows due to progressive surface waves in deep water are considered. For periodic waves, only gravity is included in the dynamic boundary condition, but both gravity and surface tension are taken into account for solitary waves. The validity of the steady first-order cnoidal wave approximation, i.e. the periodic solution of KdV, is extended to infinite depth by renormalizations. This renormalized cnoidal wave (RCW) solution is expressed as a Fourier–Padé approximation. It is analytically simpler and more accurate than fifth-order Stokes approximations. It is also capable of describing the recently discovered sharp-crested wave. A sharp-crested wave is obtained when the fluid velocity at the crest is larger than the phase speed. When the wavelength is infinite, RCW yields an algebraic solitary wave. Depending on the surface tension, the solitary wave involves one or two interfaces: a wave of depression; a wave of depression with a pocket of air; a wave of elevation with a pocket of air. Solitary waves are found for all values of the surface tension. However, this does not necessarily mean that these waves are solutions of the exact equations. Moreover, RCW approximate solitary waves always present a dipole singularity. It is also shown that a cnoidal wave in deep water can be rewritten as a periodic distribution of dipoles, each dipole representing an algebraic solitary wave. This provides a new paradigm for descriptions of water wave phenomena.

1. Introduction

Numerical schemes for solving the water wave equations are undergoing constant improvement (see Fenton 1999 for a recent review). They have successfully produced numerous valuable results. However, simplified models (Mei 1989) – such as the Korteweg–de Vries equation (KdV) or the nonlinear Schrödinger equation (NLS) – remain in common use. Indeed, highly accurate large-scale numerical simulations of complex sea states are still beyond ‘fast’ fully nonlinear models (e.g. Clamond & Grue 2001). Moreover, and more importantly, fully nonlinear numerical models provide accurate quantitative information, but they are often inefficient for describing qualitatively the mechanisms involved. On the other hand, simplified models allow such qualitative descriptions; this is a consequence of their analytical tractability.

In particular, analytical approximations of progressive waves are of paramount importance to describe more complex wave fields. For example, the concepts of rays (geometric optics) for linear waves or solitons for long nonlinear waves naturally derive from the analytical steady solutions of the simplified models. Thus, in deep water or intermediate depths, wave fields are often regarded as a superposition of (quasi-) sinusoidal waves; the nonlinearities play a secondary role, mainly acting as corrections or weak interactions. Conversely, in shallow water a wave field can be viewed as a superposition of interacting solitary waves. Such paradigms are crucial to comprehend the underlying physical mechanisms.

The search for an approximate solution is generally based on some physical considerations, depending on the wave characteristics. This has led to the development of two basic theories: the theory of short waves (Stokes waves), and the shallow-water theory. Their first-order approximations correspond to the steady solutions of NLS and KdV models, respectively. It is well known (Stoker 1957; Whitham 1974) that these theories are not uniformly valid in the complete range of water depths, from shallow to deep water. A unified description, which is also tractable, is clearly desirable (Radder 1999). A first step in this direction has been taken by renormalization of the steady KdV approximation (Clamond 1999). The improved solution consequently derived is uniformly valid from long to short waves, the wavelength being compared to the mean water depth. (In this paper the wavelength is always related to the water depth. Hence, a *short wave* is a Stokes wave, while a *long wave* is a shallow water wave.) It has also been shown that this approximation is accurate even for relatively steep waves. This solution is limited to finite depth, however. The motivation of the present work was thus to extend the validity of this solution to infinite depth.

Model equations derived from singular perturbation techniques, such as KdV and NLS, are based on approximations of the velocity potential that are not exact solutions of Laplace's equation; this is one reason for their restricted validity. For two-dimensional problems, once an approximation has been derived, its holomorphy can be forced by applying a renormalization (Clamond 1999). The approximation subsequently derived is necessarily an exact solution of Laplace's equation. The principle of renormalization is briefly described in § 2 and a generalization is given in Appendix A.

When applied to the steady solution of the KdV approximation, the renormalization yields a quite accurate solution, valid for both long and short waves. However, this renormalized cnoidal wave (RCW) approximation is not directly valid for infinite depth. For this limiting case, the corresponding solution is derived in § 3. The solution is expressed as a Fourier–Padé approximation. After obtaining the surface elevation (§ 4), and the relations between the parameters (§ 5), the cnoidal wave extension to infinite depth is completed.

In § 7, the RCW is compared with a steep exact Stokes wave and with its fifth-order classical approximation. The renormalized KdV approximation is found to be more accurate than the fifth-order one. If the speed at the crest is larger than the phase velocity (§ 8), then the RCW gives an approximation of the recently discovered *sharp-crested wave* (Lukomsky, Gandzha & Lunkomsky 2002*a*, *b*).

When the wavelength is infinite, the RCW yields an algebraic solitary wave (§ 9). For comparison with known results, the surface tension is included. For intermediate surface tensions, the RCW is in qualitative agreement with the exact known solution (Longuet-Higgins 1989). The RCW also predicts a pure gravity solitary wave and, for large surface tensions, solitary waves with a submerged bubble. The RCW solitary waves always present a dipole singularity.

In § 10, it is shown that the RCW involves a periodic distribution of dipoles, each dipole representing an algebraic solitary wave. Thus, a wave train can be viewed as a superposition of interacting algebraic solitary waves. This provides another paradigm for the description of water waves in deep water.

2. Renormalized velocity potential of a cnoidal wave in finite depth

The renormalization principle is first briefly described. For more details and generalizations see Clamond (1999) and Appendix A. The renormalization is then applied to the cnoidal wave approximation.

2.1. *Renormalization principle*

Let $\{x, y, t\}$ denote the horizontal, upward vertical and time coordinates, respectively; $y=-h$ and $y=\eta(x, t)$ are the equations of the impermeable horizontal bottom and the free surface. The most general solution of Laplace's equation, satisfying the bottom impermeability condition, for the velocity potential ϕ is

$$\phi(x, y, t) = \tfrac{1}{2}\hat{\phi}(x + \mathrm{i}y + \mathrm{i}h, t) + \tfrac{1}{2}\hat{\phi}(x - \mathrm{i}y - \mathrm{i}h, t) \tag{2.1a}$$

$$= \cos[(y + h)\partial_x]\,\hat{\phi}(x, t), \tag{2.1b}$$

where $\mathrm{i}^2 = -1$, and $\hat{\phi}(x, t) = \phi(x, -h, t)$ is the potential at the bottom. Thus, if an approximation of $\hat{\phi}$ is known, the application of (2.1) provides a holomorphic approximation of ϕ. One should note that relation (2.1) is valid for arbitrary, but finite, depth. A generalization of (2.1), valid for infinite depth too, is given in Appendix A.

2.2. *Renormalization of KdV steady solution*

The steady first-order shallow-water approximation (or KdV) is

$$\phi \approx \hat{\phi} \approx \kappa^{-1} A\ Z(\kappa\theta|m), \qquad \theta = x - Ct, \tag{2.2}$$

where κ is a kind of wavenumber, $C > 0$ is the phase speed, A is a parameter related to the maximum speed and Z is the Jacobian zeta-function of parameter m. This approximation is called a *cnoidal wave* by Korteweg & de Vries (1895), because $\hat{\phi}_x$ can be formulated with the Jacobian cn-function. The approximation (2.2) is valid for waves that have both a small amplitude a and a long wavelength L compared to the water depth. To obtain a holomorphic approximation, and thus an improved solution, the Taylor series (2.1b) cannot be used, but the renormalization formula (2.1a) yields immediately (Abramowitz & Stegun 1965, #17.4.35–36)

$$\begin{aligned}\kappa\,\phi/A &= \tfrac{1}{2}\,Z[\kappa(\theta + \mathrm{i}y + \mathrm{i}h)|m] + \tfrac{1}{2}\,Z[\kappa(\theta - \mathrm{i}y - \mathrm{i}h)|m] \\ &= Z(\kappa\theta|m) + \frac{m\,\mathrm{sn}(\kappa\theta|m)\,\mathrm{cn}(\kappa\theta|m)\,\mathrm{dn}(\kappa\theta|m)\,\mathrm{sn}^2[\kappa(y+h)|m_1]}{\mathrm{cn}^2[\kappa(y+h)|m_1] + m\,\mathrm{sn}^2(\kappa\theta|m)\,\mathrm{sn}^2[\kappa(y+h)|m_1]},\end{aligned} \tag{2.3}$$

where $m_1 = 1-m$, and $\{\mathrm{sn},\mathrm{cn},\mathrm{dn}\}$ are the elliptic functions of Jacobi. This solution is very accurate for both long and short waves (Clamond 1999). However, it is valid for finite depth only. When considering the relations between the parameters, as the parameter L/h decreases, the parameter m tends to zero. The zeta-function then tends to zero as m ($Z(x|0)=0$, $\forall x$) while the parameter A/κ tends to infinity as m^{-1}. Thus, for very deep water (say for $h > 2L$ using double-precision arithmetic), the formula (2.3) is numerically ill-conditioned and cannot be used for such depths. The corresponding formula for infinite depth must hence be derived.

3. Renormalized cnoidal wave for infinite depth

Since relation (2.3) is periodic in both the x- and y-directions, taking $h \to \infty$ is not sufficient to obtain the solution in this case. The limit must be derived including relations between parameters, i.e. A and m depend on h. This means that the conditions at the surface have to be considered. The algebraic complexity of the relations between the parameters (see Clamond 1999) makes the derivation of the limit intractable. Nevertheless, the calculation can be considerably simplified by making a simple change of parameters. The method is first demonstrated, for simplicity, with the linear theory due to Airy.

3.1. *Airy's potential from finite to infinite depth*

In the context of linearized equations, a solution of Laplace's equation and the bed boundary condition is (Airy's linear solution)

$$\phi = k^{-1}\,\mathfrak{A}\,\sin(k\theta)\cosh[k(y+h)], \tag{3.1}$$

where $\mathfrak{A}$ is an arbitrary constant while the boundary conditions at the surface have not been taken into account. As for (2.3), the limit $h \to \infty$ cannot be taken directly in (3.1), because $\mathfrak{A}$ depends on h. To obtain the potential for infinite depth, it is sufficient to introduce the change of parameter $\mathfrak{A} = \mathfrak{A}_\infty \operatorname{sech}(kh)$, and let $h \to \infty$. Thus, one obtains

$$\phi = k^{-1}\,\mathfrak{A}_\infty\,\sin(k\theta)\exp(ky), \tag{3.2}$$

which is the Airy solution for infinite depth. Hence, with a simple change of parameter, it is possible to obtain the velocity potential for infinite depth, even if the relations between parameters have not yet been derived. This operation is a renormalization too, but different from the one enforcing the holomorphy (relation (2.1)). Therefore, this renormalization will be applied to find the cnoidal wave limit in infinite depth.

3.2. *Cnoidal waves in the limit of infinite depth*

The renormalized potential (2.3) is periodic for $m \neq 1$. It is therefore expandable as a Fourier series. From the Fourier series of the zeta-function (Abramowitz & Stegun 1965, #17.4.38), we obtain the Fourier expansion of ϕ

$$\phi = \frac{\pi^2 A}{k K^2}\sum_{n=1}^{\infty} \operatorname{cosech}(n\pi K'/K)\sin(nk\theta)\cosh[nk(y+h)], \tag{3.3}$$

where $k = \pi\kappa/K$, $K = K(m)$ is the complete elliptic integral of the first kind and $K' = K(1-m)$. The Taylor expansion around $m = 0$ (Abramowitz & Stegun 1965, #17.3.21)

$$\operatorname{cosech}(n\pi K'/K) = 2\,(m/16)^n + O(m^{n+2}), \tag{3.4}$$

shows that m plays a role in (2.3) similar to the one that $\mathfrak{A}$ plays in Airy's solution. This analogy suggests the change of parameters

$$m = 16\, m_\infty \operatorname{sech}(kh), \qquad A = K^2 A_\infty /\, 2\pi^2 m_\infty. \tag{3.5a, b}$$

Thus, taking $h \to \infty$, the potential (3.3) becomes

$$\frac{k\,\phi}{A} = \sum_{n=1}^{\infty} m^{n-1}\sin(nk\theta)\exp(nky) \tag{3.6a}$$

$$= \frac{\sin(k\theta)\exp(ky)}{1 - 2m\cos(k\theta)\exp(ky) + m^2\exp(2ky)}, \tag{3.6b}$$

where the indices '∞' have been omitted for brevity. The relation (3.6) is the renormalized cnoidal wave (RCW) velocity potential for infinite depth. Since it is the limit of a cnoidal wave, it can be called a 'cnoidal wave in infinite depth'.

Similarly, the stream function is

$$\frac{k\,\psi}{A} = \frac{\cos(k\theta)\exp(ky) - m\exp(2ky)}{1 - 2\,m\cos(k\theta)\exp(ky) + m^2\exp(2ky)}. \tag{3.7}$$

Clearly, relations (3.6)–(3.7) satisfy exactly the Laplace equation and $\nabla\phi \to 0$, $\psi \to 0$ as $y \to -\infty$. If $m = 0$, the Airy solution is recovered. If $m \neq 0$, the Fourier–Padé approximation (3.6) involves an infinite number of harmonics. This should be compared with the short-wave theory (Stokes expansion) where non-zero harmonics are found at fourth-order only (see (7.1)); their derivations require a considerable amount of algebra. One can then expect the RCW to be an accurate approximation (see § 7).

The surface elevation and relations between parameters provided by the shallow-water theory are inconsistent after renormalization. They must therefore be rederived.

4. Surface elevation and related parameters

The mass conservation equation ($\eta_t + \tilde{\psi}_x = 0$ with $\tilde{\psi} = \psi|_{y=\eta}$) can be integrated, for a progressive wave and with relation (3.7), to give

$$\eta = \frac{\tilde{\psi}}{C} - \alpha = \frac{A}{k\,C}\frac{\cos(k\theta)\exp(k\eta) - m\exp(2k\eta)}{1 - 2\,m\cos(k\theta)\exp(k\eta) + m^2\exp(2k\eta)} - \alpha, \tag{4.1}$$

where α is an integration constant. η being chosen as the surface elevation from rest $y = 0$, we impose $\langle\eta\rangle \equiv (k/2\pi)\int_{-\pi/k}^{\pi/k} \eta\,\mathrm{d}\theta = 0$, giving an equation for α:

$$\alpha = \langle\tilde{\psi}/C\rangle. \tag{4.2}$$

Note that relation (4.1) gives an implicit definition of $\eta(\theta)$, but $\theta(\eta)$ is explicit, i.e.

$$\cos(k\theta) = \frac{mA/C \;+\; k(\eta+\alpha)\,[\exp(-2k\eta) + m^2]}{A/C \;+\; 2mk(\eta+\alpha)}\exp(k\eta). \tag{4.3}$$

Let a denote the elevation of the crest above the still water level, then

$$ka \equiv k\eta(0) = \frac{A/C}{\mathrm{e}^{-ka} - m} - k\alpha, \tag{4.4}$$

and therefore

$$A/C = k(a+\alpha)\,(\mathrm{e}^{-ka} - m). \tag{4.5}$$

Let b denote the depth of the trough below the still water level, then

$$kb \equiv -k\eta(\pi/k) = \frac{A/C}{\mathrm{e}^{kb} + m} + k\alpha, \tag{4.6}$$

and the wave height is $H = a + b$. Eliminating A/C between (4.4) and (4.6), one obtains

$$k\alpha = -ka + kH(m\,\mathrm{e}^{ka} + \mathrm{e}^{kH})(1 + \mathrm{e}^{kH})^{-1}. \tag{4.7}$$

To complete the solution, two more relations between the parameters must be obtained.

5. Relations between parameters

The dynamic boundary condition remains to be considered. At the free surface, the pressure is zero and the surface tension is neglected. Then, for a periodic progressive gravity wave observed in the frame of reference where $\nabla\phi \to 0$ as $y \to -\infty$, the Bernoulli equation applied at the surface yields

$$g\eta - C\,\tilde{u} + \tfrac{1}{2}\,\tilde{u}^2 + \tfrac{1}{2}\,\tilde{v}^2 = 0, \tag{5.1}$$

where $\tilde{u} = \phi_x|_{y=\eta}$, $\tilde{v} = \phi_y|_{y=\eta}$ and g is the acceleration due to gravity. Unfortunately, the RCW is not an exact solution of (5.1). Nevertheless, the parameters can be chosen in a way that solves this equation approximately. The collocation procedure described by Clamond (1999) gives a very simple and accurate method to derive the relations sought.

Applied at the crest, where $\tilde{u} = A\,\mathrm{e}^{ka}[1 - m\mathrm{e}^{ka}]^{-2}$, the Bernoulli equation (5.1) yields

$$2ga(1 - m\,\mathrm{e}^{ka})^4 - 2CA\,\mathrm{e}^{ka}(1 - m\,\mathrm{e}^{ka})^2 + A^2\,\mathrm{e}^{2ka} = 0. \tag{5.2}$$

Similarly, applied at the trough, where $\tilde{u} = -A\,\mathrm{e}^{-kb}[1 + m\mathrm{e}^{-kb}]^{-2}$, (5.1) yields

$$2gb(1 + m\,\mathrm{e}^{-kb})^4 - 2CA\,\mathrm{e}^{-kb}(1 + m\,\mathrm{e}^{-kb})^2 - A^2\,\mathrm{e}^{2kb} = 0. \tag{5.3}$$

The RCW approximation is now complete. The parameters g, k and a being chosen, relations (4.2), (4.5), (4.6), (5.2) and (5.3) provide implicit definitions of α, A, b, C and m. This system of equations can be solved numerically using Newton's method, for example. The integration involved in (4.2) can be computed via the trapezoidal formula, which is of infinite-order for periodic functions.

6. Remarks

The collocation method presented above is open to the criticism that the choice of the collocation nodes is somewhat arbitrary. Many other methods can also be used for finding approximate relations between parameters.

The elegant average Lagrangian method (Whitham 1974) leads to intractable algebra. The Lagrangian must hence be minimized via some numerical procedures. A similar technique consists of minimizing numerically the error $\|g\eta - C\tilde{u} + \frac{1}{2}\tilde{u}^2 + \frac{1}{2}\tilde{v}^2\|$. These methods require considerable amounts of computation, and some tests have shown that their accuracies are comparable to the collocation method. They are therefore not best for practical purposes.

Another method has been used by McCowan (1891) for the solitary wave, and may be employed here too. θ being an explicit function of η, the Bernoulli equation (5.1) can be rewritten as a function of η only. McCowan's method then consists in approximatively solving this equation and expanding in a power-series in η. Setting to zero the coefficients of the lower-order terms η^n gives relations defining C and m. This method has several drawbacks, however. First, it involves much algebra. Secondly, expanding the Bernoulli equation in power-series in η is arbitrary, and it is efficient only if η is small. Many other types of expansions could be preferred. Thirdly, the renormalizations of various approximations do not lead to θ as an explicit function of η, in general. McCowan's method cannot be applied in such cases.

Some well-known exact integral relations (Starr 1947) could also be used as closure equations. The integral quantities have to be computed numerically, and there are a limited number of them. Thus, this method is not convenient and cannot be employed if many relations have to be obtained.

Alternatively, η can be defined by solving exactly the Bernoulli equation (5.1), and the relations between parameters can be obtained by solving approximately the mass conservation (4.1). This leads to more complicated algebra for a comparable accuracy.

Therefore, considering analytical and numerical simplicities, as well as accuracy, the collocation method appears to be the 'best'. For renormalizations involving more than two free parameters, the collocation points may be chosen equally spaced in $a \geqslant \eta \geqslant -b$, or at the Gauß–Lobatto nodes, for examples. This is what pseudo-spectral numerical models involve, where the equations are solved exactly at selected points. With the RCW, only two computational nodes are required, since there are only two degrees of freedom.

7. Stokes wave

The RCW approximation is compared with an exact numerical solution (Fenton 1988) and with a fifth-order approximation (FOA) (Fenton 1990):

$$k\sqrt{k/g}\ \phi = \epsilon\left(1 - \tfrac{1}{2}\epsilon^2 - \tfrac{37}{24}\epsilon^4\right)\exp(ky)\sin(k\theta) + \tfrac{1}{2}\epsilon^4\exp(2ky)\sin(2k\theta) + \tfrac{1}{12}\epsilon^5\exp(3ky)\sin(3k\theta) + O(\epsilon^6), \quad (7.1)$$

$$k\,\eta = \epsilon\left(1 - \tfrac{3}{8}\epsilon^2 - \tfrac{211}{192}\epsilon^4\right)\cos(k\theta) + \tfrac{1}{2}\epsilon^2\left(1 + \tfrac{2}{3}\epsilon^2\right)\cos(2k\theta) + \tfrac{3}{8}\epsilon^3\left(1 + \tfrac{33}{16}\epsilon^2\right)\cos(3k\theta) + \tfrac{1}{3}\epsilon^4\cos(4k\theta) + \tfrac{125}{384}\epsilon^5\cos(5k\theta) + O(\epsilon^6), \quad (7.2)$$

$$\sqrt{k/g}\ C = 1 + \tfrac{1}{2}\epsilon^2 + \tfrac{1}{8}\epsilon^4 + O(\epsilon^6), \quad (7.3)$$

where $\epsilon = \frac{1}{2}kH$ is a wave steepness.

The accuracy of the approximations is investigated by considering an exact steep wave with $\epsilon = 0.4241$ ($ka = 0.5547$, $kb = 0.2936$), which is the highest wave computable with Fenton's program. Of course, all the solutions (RCW, FOA and exact) are compared in the same frame of reference (such as $\nabla\phi \to 0$ as $y \to -\infty$), with the same reference for the water level (such as η averages to zero), and with the same wavelength. These three quantities being fixed, only one more parameter can be taken equal for all the solutions.

Comparing the surface elevations with equal heights (figure 1*a*), with equal amplitudes (figure 1*b*) and with equal trough depths (figure 1*c*), the RCW is more accurate than the FOA, in every case. The overall agreement between the exact and approximate surface elevations is better when comparing with equal heights. Therefore, we compare some other quantities with the same parameter ϵ.

Considering the horizontal and vertical velocities at the surface (figure 2), as well as the horizontal velocity under the crest (figure 3), the RCW is more accurate than the FOA. Quantitatively, the relative errors of the RCW and FOA, for $\epsilon = 0.4241$, are respectively:

(i) 0.5% and 3.4% for the wave amplitude;
(ii) 0.9% and 6.5% for the trough height;
(iii) 5.4% and 8.9% for the speed at crest;
(iv) 0.2% and 3.6% for the speed at trough;
(v) 6.5% and 7.0% for the maximum vertical velocity at the surface.

Of course, the RCW accuracy is higher for smaller waves. The FOA matches the RCW only for the maximum vertical velocity at the surface. However, RCW has a better overall vertical velocity profile (figure 2). Note that the good accuracy of RCW at the crest and trough is partly due to the choice of the collocation nodes. Other choices may increase the RCW accuracy at other positions.

Its simplicity and accuracy make the RCW an attractive alternative to FOA for practical uses. Moreover, investigating waves close to the highest one with the FOA is

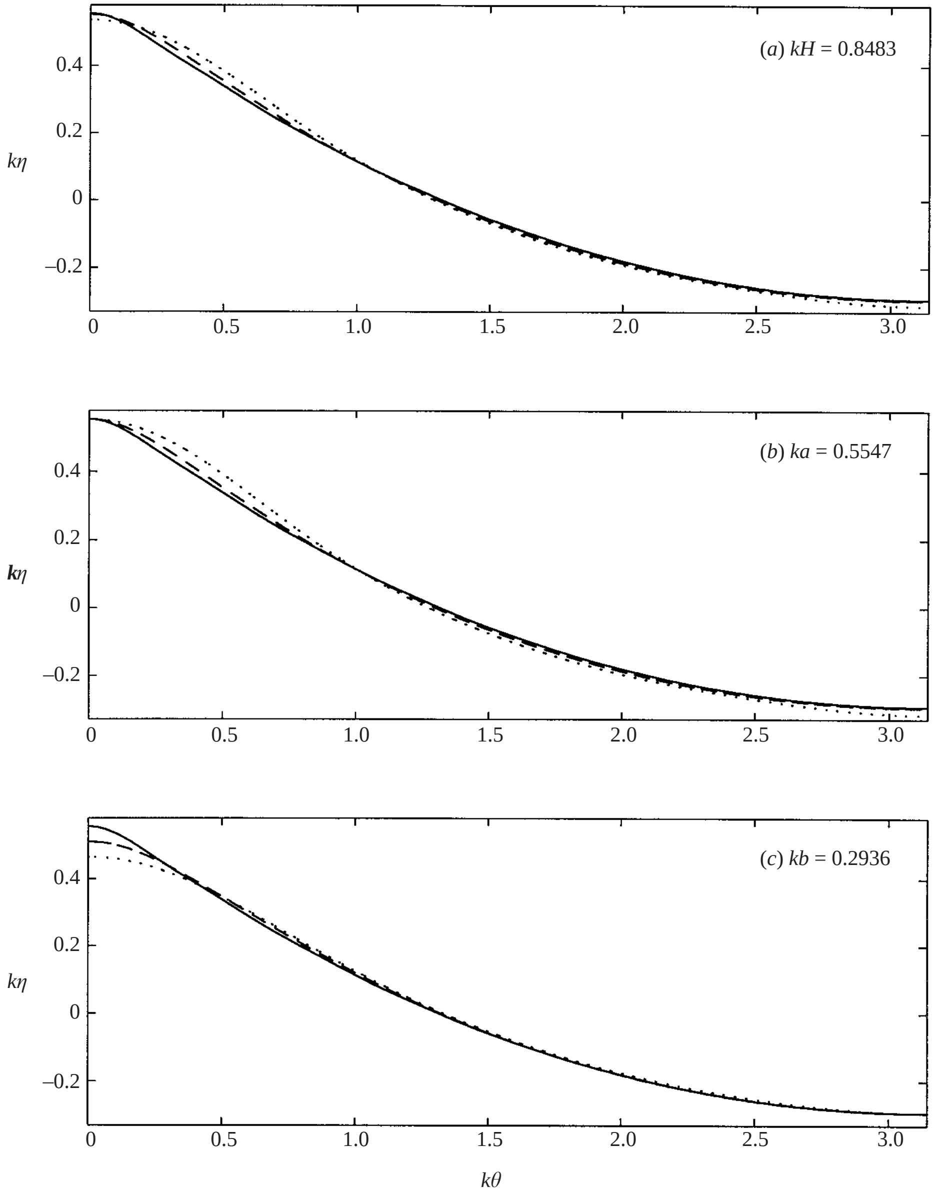

FIGURE 1. Surface elevation of a steep Stokes wave ($\epsilon = 0.4241$): —, exact; – –, renormalized cnoidal wave; · · ·, fifth-order approximation. (*a*) Equal heights; (*b*) equal amplitudes; (*c*) equal trough depths.

irrelevant, but the RCW provides some good results, as we shall see in the following section.

8. Higher renormalized cnoidal waves

For small and moderately steep waves, the wave parameters increase as ka increases. On the other hand, the variation of some parameters ceases to be monotonic for steep waves. In particular, this is the case for the parameter ϵ (figure 4); therefore a large

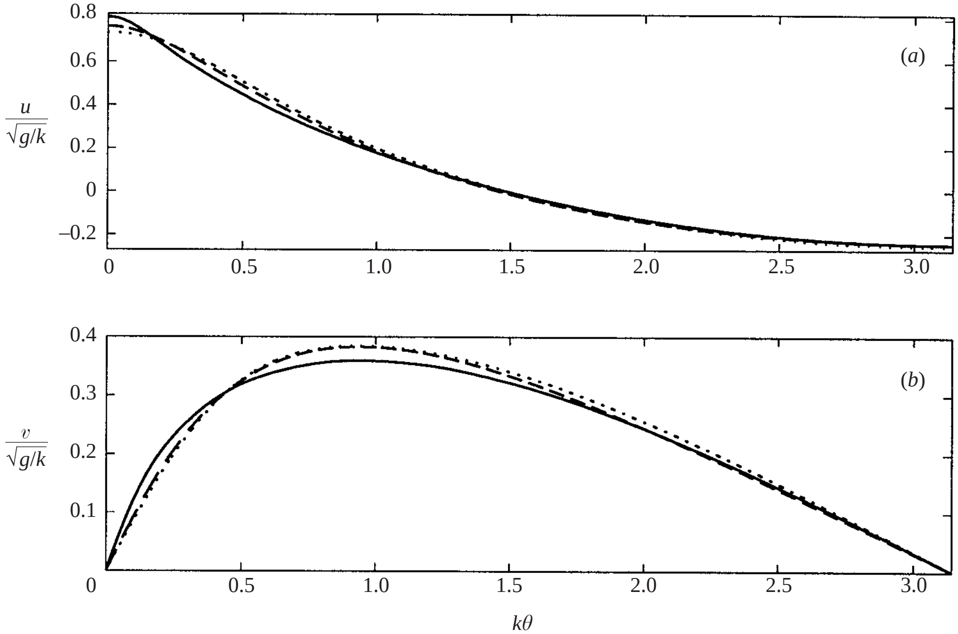

FIGURE 2. Velocities at the surface of a steep Stokes wave ($\epsilon = 0.4241$): —, exact; – –, renormalized cnoidal wave; $\cdots$, fifth-order approximation. (*a*) Horizontal velocity; (*b*) vertical velocity.

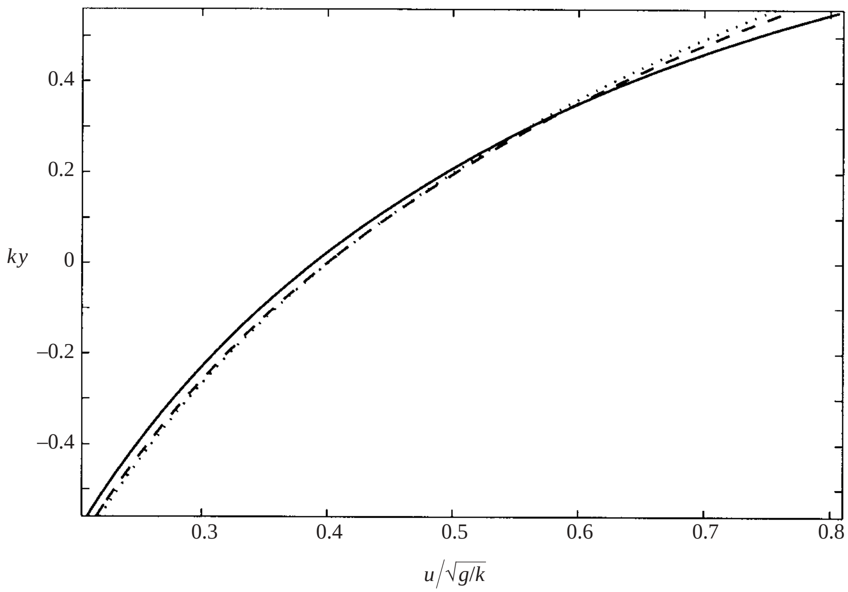

FIGURE 3. Velocity under the crest of a steep Stokes wave ($\epsilon = 0.4241$): —, exact; – –, renormalized cnoidal wave; $\cdots$, fifth-order approximation.

wave is not uniquely defined by ϵ. Thus, the parameter ka – that is also a steepness parameter – appears to be a 'better' parameter than ϵ to characterize a wave (at least for RCW).

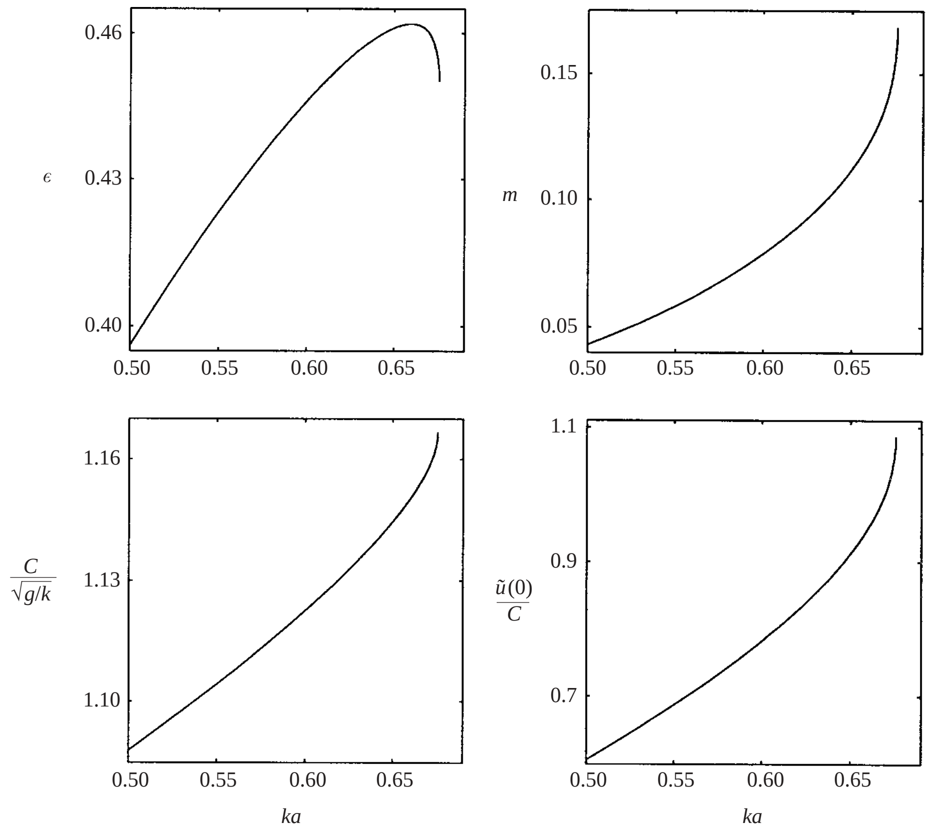

FIGURE 4. Relations between some of the RCW parameters.

We investigate here three interesting cases: the wave of maximum height, the corner flow and the sharp-crested wave.

8.1. *Wave of greatest height*

The maximum height ($\epsilon \approx 0.4619$) is obtained for $ka \approx 0.6596$, for which $C - \tilde{u}(0) \approx 0.06\sqrt{g/k}$. Thus, according to the RCW solution, the wave of greatest height is not the one forming a corner at the crest; this solution occurs for a larger amplitude. The RCW of maximum height is found for a steepness that exceeds significantly the maximum exact one ($\epsilon \approx 0.4432$). Nevertheless, the RCW solution remains qualitatively consistent for such a large wave.

Indeed, for an exact wave, the streamlines (in the frame of reference where the flow is steady) intersect at a right-angle above the crest, forming a stagnation point (Grant 1973; Longuet-Higgins & Fox 1977, 1978; Lukomsky *et al.* 2002*b*). The stagnation point approches the surface as ka increases. The RCW presents this characteristic (figure 5) in agreement with the previous works.

8.2. *Corner flow*

When the fluid velocity at crest equals the phase velocity ($C^2 = 2ga$), the free surface forms a corner, that is a stagnation point. The RCW gives this type of solution for $ka \approx 0.6694$ ($\epsilon \approx 0.4602$) but the crest forms a right-angle, whereas the exact angle must be $2\pi/3$. This discrepency can be easily explained.

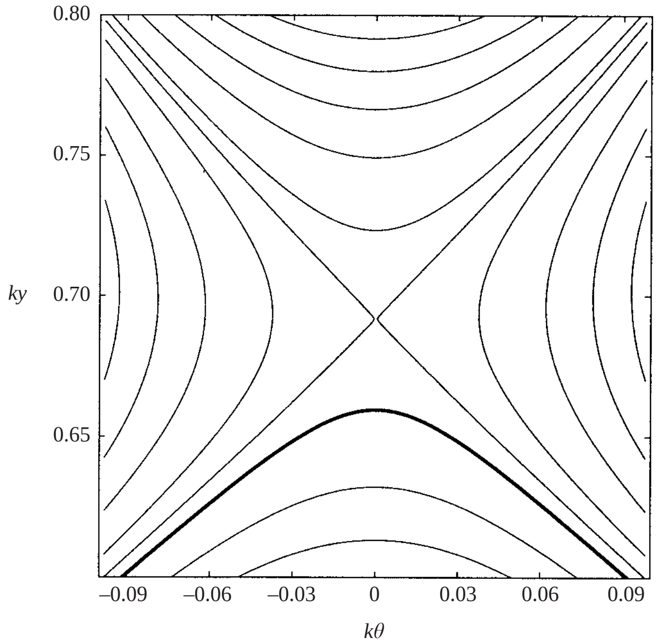

FIGURE 5. Streamlines of the RCW of greatest height ($ka = 0.66$). The free surface is shown with a bold line.

Grant (1973) predicted that the exact solution involves several dipole singularities above the crest. A dipole singularity generates a stagnation point forming a right-angle between the streamlines. Grant suggested that the only continuous approach to the 120° angle is for coalescing singularities. This hypothesis has been supported by Longuet-Higgins & Fox (1977, 1978).

The RCW involves only one singularity above the crest (see § 10). Thus, no merging between several singularities can take place. Therefore, at the stagnation point, the streamlines intersect at a right-angle for all amplitudes, including the corner flow.

8.3. *Sharp-crested wave*

Taking $\tilde{u}(0) > C$ (but $C^2 > 2ga$), Lukomsky *et al.* (2002*a*, *b*) recently discovered a new type of solution which they named *sharp-crested waves*. A sharp-crested wave has a stagnation point below the surface: it is a singular solution. For this reason, the computation of an exact sharp-crested wave is very demanding. Lukomsky *et al.* developed an *ad hoc* numerical scheme involving arithmetic with ninety-six decimal digits!

The RCW approximation admits sharp-crested wave solutions when $ka > 0.6694$. The maximum amplitude for the RCW sharp-crested wave is $ka \approx 0.6756$ ($\epsilon \approx 0.4502$). Of course, these values are not quantitatively correct; the main interest of the RCW, for sharp-crested waves, is its ability to predict their possible existence. Qualitatively, the sharp-crested RCW admits a stagnation point forming a right-angle below the crest (figure 6), in agreement with the results of Lukomsky *et al.* (2002*b*).

According to the RCW solution, a sharp-crested wave has a larger amplitude and is steeper than a Stokes wave of the same height (figure 7). Compared to the exact solution of Lukomsky *et al.* (2002*a*), the RCW approximation exaggerates the differences between sharp-crested and Stokes waves of the same height.

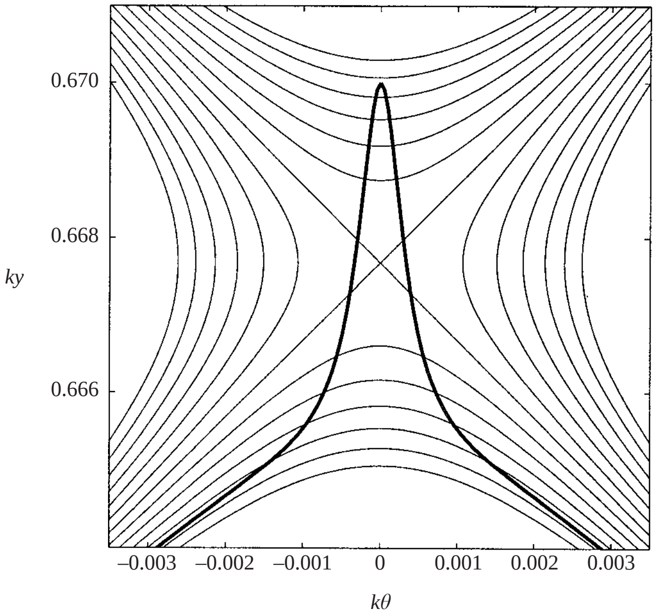

FIGURE 6. Streamlines of a sharp-crested RCW ($ka = 0.67$). The free surface is shown with a bold line.

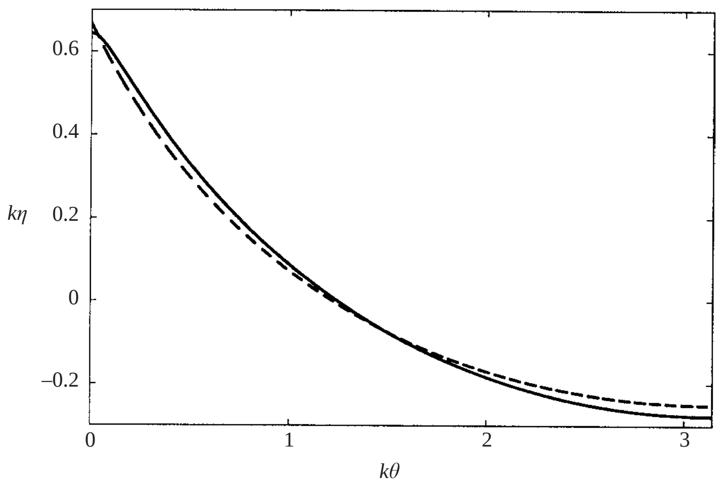

FIGURE 7. A sharp-crested RCW versus a regular RCW of the same height ($\epsilon = 0.46$): – –, sharp-crested wave ($ka = 0.67$); —, Stokes wave ($ka = 0.643$).

Though convincing, the numerical study of Lukomsky *et al.* is not a mathematical proof of the existence of the sharp-crested wave. The RCW provides an independant confirmation of the probable existence of this kind of solution. A sharp-crested wave, if it exists, is certainly unstable. It is therefore very unlikely that it will be observed in an experimental wave tank and, *a fortiori*, at sea. Its interest is more theoretical than practical.

9. Solitary wave

We consider here the RCW approximation as the wavelength tends to infinity. Solitary waves in deep water have been found when the surface tension is taken into account (Longuet-Higgins 1989). Since Korteweg & de Vries (1895) have included surface tension in their equation, it is consistent to use the RCW for investigating capillary–gravity waves. We shall see that the RCW admits a solitary wave solution for all values of the surface tension. It must be emphasized that this result does not constitute a mathematical proof of the existence of such waves. The main goal of this section is to demonstrate the usefullness of the renormalization for finding new possible solutions that are not included in the original approximation.

9.1. *Velocity potential, stream function and surface elevation*

Introducing the change of parameters $A = k^2\ell^2 A^\star$ and $m = 1 + k\ell$, then taking $k \to 0$ (yet another renormalization), the RCW relations (3.6) and (3.7) become

$$\frac{\phi}{\ell A} = \frac{\ell\,\theta}{\theta^2 + (y+\ell)^2}, \qquad \frac{\psi}{\ell A} = \frac{-\ell\,(y+\ell)}{\theta^2 + (y+\ell)^2}, \tag{9.1a,b}$$

where the $\star$ on A has been omitted for brevity. Since this non-periodic type of solution involves algebraic functions, it is generally called an *algebraic solitary wave*. This flow corresponds to a dipole localized at $\theta = 0$ and $y = -\ell$. The parameter ℓ can be viewed as a characteristic 'wavelength'.

The mass conservation (4.1) here gives the implicit definition of η:

$$\eta = -\frac{A}{C}\,\frac{\ell^2(\eta+\ell)}{\theta^2 + (\eta+\ell)^2}, \tag{9.2}$$

where the condition $\eta(\infty) = 0$ has been considered. The value of η is obtained explicitly by solving a third-order polynomial equation, but it is simpler to consider θ as a function of η, i.e.

$$\theta^2 = -(\eta+\ell)[\eta + \ell + \ell^2 A/C\eta]. \tag{9.3}$$

From (9.3), depending on the ratio A/C, η has one or two branches of solutions: a main one extending to infinity representing the free surface, and a secondary closed one representing a 'bubble'. The dipole is part of one of these interfaces. In the neighbourhood of the dipole, although the velocity field is singular, η is a regular function defined by continuity using (9.3).

At the origin $\eta(0) = \eta_0$, the definition of the surface (9.2) gives

$$\eta_0/\ell = -1, \qquad \eta_0/\ell = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1 - 4A/C}. \tag{9.4}$$

The last two roots are real only if $C \geqslant 4A$. Thus, for $C < 4A$ only the free surface is present, for $C > 4A$ the solution involves a free surface and a submerged bubble, the two interfaces having a common point for $C = 4A$.

9.2. *Relations between parameters*

When the surface tension is taken into account, and for a progressive solitary wave observed in the frame of reference where $\tilde{u}(\infty) = 0$, the Bernoulli equation applied at the free surface gives

$$g\eta - C\,\tilde{u} + \tfrac{1}{2}\tilde{u}^2 + \tfrac{1}{2}\,\tilde{v}^2 - \tau\,\eta_{\theta\theta}\left(1 + \eta_\theta^2\right)^{-3/2} = 0, \tag{9.5}$$

where τ is the surface tension coefficient. Due to the presence of the dipole, the Bernoulli equation is singular at $\theta = 0$. Therefore, the collocation method used for the periodic waves cannot be employed. Instead, since we are dealing with algebraic expressions, we shall rewrite the Bernoulli equation as a polynomial in η and set to zero the coefficients of the lowest-order powers of η (McCowan's method). As η is small for large θ, this method corresponds to an asymptotic resolution of the equation.

Using the explicit relation (9.3), one obtains

$$\frac{\tilde{u}}{C} = 2\frac{C}{A}\frac{\eta^2}{\ell^2} + \frac{\eta}{\eta+\ell}, \qquad \frac{\tilde{u}^2+\tilde{v}^2}{C^2} = \left(\frac{\eta}{\eta+\ell}\right)^2, \tag{9.6a,b}$$

$$\frac{\eta_{\theta\theta}}{\left(1+\eta_\theta^2\right)^{3/2}} = -\frac{\mathrm{d}}{\mathrm{d}\eta}\left\{\frac{\ell^2 - 2\eta^2(\eta+\ell)C/A\ell}{[\ell^4 - 4\eta^2(\eta+\ell)^2 C/A]^{1/2}}\right\}. \tag{9.7}$$

Then, after one integration with respect of η and considering the condition at $\theta=\infty$, the Bernoulli equation (9.5) yields

$$\frac{\eta}{\eta+\ell} - \frac{\eta}{\ell} + \frac{g\eta^2}{\ell C^2} - \frac{4C\eta^3}{3A\ell^3} + \frac{2\tau}{\ell C^2}\frac{\ell^2 - 2\eta^2(\eta+\ell)C/A\ell}{[\ell^4 - 4\eta^2(\eta+\ell)^2 C/A]^{1/2}} = \frac{2\tau}{\ell C^2}. \tag{9.8}$$

The integration is performed because it yields a simpler equation. After some elementary algebra, equation (9.8) can be rewritten as a polynomial in η. Hence, setting to zero the coefficient of the lowest-order term in η, one obtains

$$C^2 = g\ell. \tag{9.9}$$

Therefore $\ell \geqslant 0$, meaning that the dipole is always below the still water level. Setting to zero the coefficient of the following term gives

$$A/C = 4\left(\tfrac{1}{3} - B\right), \tag{9.10}$$

where $B = \tau/g\ell^2$ is a Bond number. The RCW approximation for the algebraic solitary wave is now complete.

9.3. *Analysis of RCW solitary waves*

For $B < \frac{13}{48}$, the solution involves a free surface only. For $B=0$, the RCW solution yields a pure gravity solitary surface wave (figure 8*a*). As B increases, the trough becomes more narrow (figure 8*b*), becoming a 'pocket of air' (figure 8*c*) that is closed for $B=\frac{13}{48}$ (figure 8*d*). Note that the radius of curvature at the lowest point of the surface, i.e.

$$R_0 \equiv \left[\frac{\eta_{\theta\theta}}{\left(1+\eta_\theta^2\right)^{3/2}}\right]^{-1}_{\eta=-\ell} = 2\ell\left(\tfrac{1}{3} - B\right), \tag{9.11}$$

is never infinite, nor zero, for $B < \frac{13}{48}$. For $B = \frac{13}{48}$ the surface self-intersects at a right-angle corresponding to a stagnation point. Numerical resolution of the exact equations (Longuet-Higgins 1989; Vanden-Broeck & Dias 1992) suggest that the angle should instead be zero.

For $\frac{13}{48} < B < \frac{1}{3}$, the surface is still a wave of depression, but below the surface there is a submerged 'bubble' (figure 8*e*). The distance between the surface and the bubble increases as B increases. At the same time, the magnitude of the surface depression and the 'height' of the bubble decrease, vanishing for $B=\frac{1}{3}$.

For $B > \frac{1}{3}$, the surface is a wave of elevation with a submerged bubble (figure 8*f*). The amplitude of the surface and the size of the bubble increase as B increases.

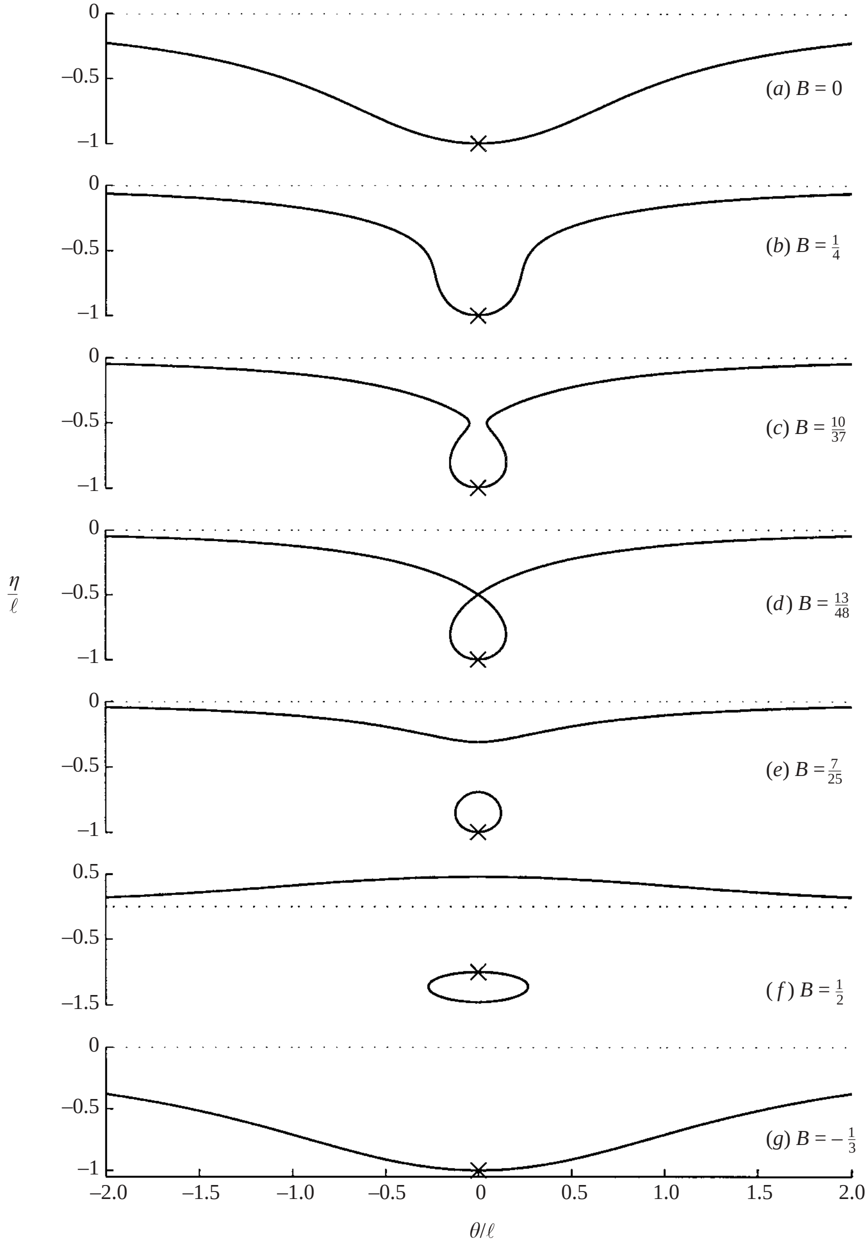

FIGURE 8. Renormalized solitary waves in deep water: —, interfaces; · · ·, still water level; ×, dipole's position.

Finally, for a negative surface tension coefficient (for fluids other than air and water) the Bond number is negative. The solution involves a wave of depression. As B decreases the wave becomes broader (figure 8*g*).

The RCW provides an approximation for solitary waves in deep water. These solutions are all singular. This does not mean that exact regular solitary waves do not exist. Indeed, the RCW is just a simple approximation. Exact regular solitary waves can exist if the singularity is above the surface or inside the bubble. Allowing a singularity at the interface, or within the fluid, exact singular solitary waves may also exist.

9.4. *Discussion*

Including surface tension, Longuet-Higgins (1989) obtains numerically an 'exact' solitary wave in deep water. He finds no singularity at the surface, the singularity being above it. He observes an algebraic decay similar to the RCW in the far field, but around the origin, the exact solution is more complex. Longuet-Higgins obtains solitary waves for intermediate Bond number, corresponding to the RCW solutions on figures 8(b, c, d). Compared to the exact solution, the RCW is not quantitatively accurate but is qualitatively reasonably correct.

Due to the presence of a singularity close to the surface, Longuet-Higgins could not decrease B below a certain limit. This does not necessarily mean that the solution does not exist. It is possible, for smaller surface tension, that the singularity is so close to the surface that Longuet-Higgins' numerical scheme becomes inefficient. Thus, as for the sharp-crested wave, the computation of exact capillary–gravity solitary waves, with very small surface tension, may be so demanding that very high-precision arithmetic is required. With such an improved scheme, the surface tension may be significantly decreased. The (non) existence of the solitary wave in deep water might then be 'demonstrated' for small B.

Iooss & Kirrmann (1996) proved the existence of solitary capillary–gravity waves in deep water that decay at least as $|\theta|^{-1}$. They do not indicate a minimum value for the Bond number.

In finite depth, Lavrantiev (1946) proved the existence of a solitary gravity wave as the limit of a periodic wave, when the wavelength tends to infinity. He also proved the non-existence of a solitary wave of depression. His method can probably be extended to infinite depth, implying that a (regular) solitary gravity wave cannot exist in deep water. But, allowing a singular flow at the surface, such solution may exist.

10. Dipole expansions

Simplicity and accuracy are not the only interesting features of the RCW. It also provides qualitative insight, as already seen by its capability of describing sharp-crested and solitary waves. More insights are obtained by considering the singularities of the RCW.

The complex velocity potential associated with the RCW:

$$f(z) \equiv \phi + \mathrm{i}\psi = \frac{\mathrm{i}A/k}{\exp(\mathrm{i}kz) - m}, \qquad z = x - Ct + \mathrm{i}y, \tag{10.1}$$

has an infinite number of simple poles at $kz = 2n\pi - \mathrm{i}\log(m)$, $n = 0, \pm 1, \pm 2, \ldots$, with residues A/mk^2. These singularities are above the free surface if $ka < -\log(m)$, a condition less stringent than $\tilde{u}(0) \leqslant C$. Sharp-crested waves are therefore found for $\tilde{u}(0) > C$ and $ka < -\log(m)$. For the sharp-crested wave of maximum amplitude $ka \approx 0.6756$ and $\log m^{-1} \approx 1.78$. Therefore, the singularity never 'touches' the surface.

Using the Mittag–Leffler theorem (Henrici 1974, § 7.10), (10.1) can be rewritten

$$\frac{mkf}{A} = \frac{\mathrm{i}m}{1-m} + \sum_{n=-\infty}^{+\infty} \frac{1}{kz - 2n\pi + \mathrm{i}\log(m)} + \frac{1}{2n\pi - \mathrm{i}\log(m)}. \tag{10.2}$$

This potential represents a periodic distribution, in the x-direction, of dipoles of equal intensities. Each dipole generates an algebraic solitary wave (see § 9). Hence, a periodic wave in infinite depth can be viewed as a superposition of algebraic solitary waves, at least in a first (but accurate) approximation. Similarly, a cnoidal wave in finite depth can be viewed as a periodic distribution, in the x-direction, of classical solitary waves, as is well-known (see e.g. Whitham 1984). Note that an algebraic solitary wave is the limit as $h \to \infty$ of a classical solitary wave (Appendix B).

Water waves are often regarded as a superposition of (quasi) sinusoidal waves, especially in deep water or intermediate depth. With this paradigm, the nonlinearities play a minor role, mainly acting as corrections or secondary interactions. It has also been suggested that wave field can be viewed as interacting envelope solitons (solitonic turbulence, Zakharov *et al*. 1988). The new paradigm, suggested by the RCW, is that a wave field can always be regarded as a superposition of solitary waves, no matter what the depth. This paradigm differs from the solitonic turbulence in the sense that it concerns the 'carrier wave' and not its envelope. All these paradigms provide different qualitative descriptions of the water wave problem, the 'best' viewpoint depending on the problem.

There is an obvious analogy in quantum mechanics: the wave–particle duality. The interacting solitary waves paradigm for deep water remains valid even if the individual solitary wave does not exist. Indeed, in nuclear physics for example, there are elementary particles (e.g. proton) composed of more elementary particles (e.g. quarks) that cannot exist separately.

Algebraic solitary waves are generated by dipoles. Dipoles are composed of two vortices (or sinks) of same the intensity and opposite signs. An algebraic solitary wave can therefore be decomposed into further elementary solutions. These more 'fundamental' solitary waves have already been studied (Lukomsky 1994; Lukomsky & Sedletsky 1994).

11. Conclusion

Applying renormalizations to the classical cnoidal wave approximation, we have extended its validity to infinite depth. This improved approximation (RCW) appeared to be analytically simple and accurate, even for large waves. It is an attractive alternative to the high-order Stokes expansions for practical purposes.

For very steep waves, the RCW solution is not as accurate, but is qualitatively consistent. If the fluid velocity at crest exceeds the phase velocity, the RCW describes the recently discovered sharp-crested wave (Lukomsky *et al*. 2002*a*, *b*). With the RCW approximation the sharp-crested waves are obtained more easily than by the very involved numerical scheme of Lukomsky *et al*. Thus, renormalization provides a useful tool to derive more efficient numerical schemes. For example, one can seek accurate solutions as power-series expansions of the fundamental function (10.1). This is what Lukomsky & Gandzha (2003) attempt for investigating sharp-crested waves in a more efficient way. They obtained this expansion by other means than renormalization, however. This expansion can be viewed as the deep-water limit of the renormalized high-order cnoidal wave expansion (Fenton 1979).

When the wavelength is infinite, the RCW reduces to an algebraic solitary wave. This wave corresponds to a flow generated by a dipole at the surface; this is therefore a singular solution. To compare with some known results on solitary capillary–gravity waves in deep water, the surface tension was taken into account. Though not accurate, the RCW appears physically relevant, at least for intermediate Bond numbers. Moreover, the RCW suggests the possible existence (as solutions of the exact equations) of solitary waves for large, small and negative Bond number. *Ad hoc* numerical investigations may support to this conjecture.

Considering the singularities, we found that any (steady) deep-water wave field can be interpreted as superposition of interacting solitary waves, at least in a first accurate approximation. This provides a new paradigm for interpreting wave phenomena in deep water. This paradigm is universal in the sense that it can also be used for intermediate depths and shallow water. There is an analogy with wave–particle duality in quantum mechanics. These (algebraic) solitary waves are generated by dipoles. One can then imagine an efficient numerical model, based on dipole expansion, for simulating complex (unsteady) sea states.

With the help of renormalization, the validity of the shallow-water approximation has been extended considerably: from shallow to deep water, from infinitesimal to finite amplitudes. Most of the characteristics of a steady wave are captured by the renormalization procedure. Conversely, solutions obtained in deep water, from the nonlinear Schrödinger equation for example, may be efficient for describing waves in shallow water, after renormalization. This can be achieved from the generalized renormalization principle described in Appendix A.

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Appendix A. Renormalization principle for deep water

For potential and two-dimensional flows, the Cauchy–Riemann relations $\psi_x = -\phi_y$, $\psi_y = \phi_x$ hold, $\{\phi;\psi\}$ being the velocity potential and the stream function. Denoting with 'hats' the quantities at the (horizontal) bottom $y = -h$, the bottom impermeability yields $\hat{\psi} = 0$. The most general solution of these equations is

$$\phi(x, y, t) = \cos[(y+h)\,\partial_x]\,\hat{\phi}(x, t) = \tfrac{1}{2}\,\hat{\phi}(z + \mathrm{i}h, t) + \tfrac{1}{2}\,\hat{\phi}(z^* - \mathrm{i}h, t), \qquad \text{(A 1)}$$

$$\psi(x, y, t) = \sin[(y+h)\,\partial_x]\,\hat{\phi}(x, t) = \tfrac{1}{2\mathrm{i}}\,\hat{\phi}(z + \mathrm{i}h, t) - \tfrac{1}{2\mathrm{i}}\,\hat{\phi}(z^* - \mathrm{i}h, t), \qquad \text{(A 2)}$$

where $z = x + \mathrm{i}y$, $z^* = x - \mathrm{i}y$ and $\mathrm{i}^2 = -1$. Formulae (A 1)–(A 2) are valid for arbitrary, but finite, depth. Using the potential at the bottom is not the only possibility and it is not convenient for infinite depth. For example, one can also use the potential at the still water level $y = 0$. Denoting with 'overbars' the quantities at $y = 0$, the Taylor expansions (A 1)–(A 2) give $\bar{\phi} = \cos(h\partial_x)\,\hat{\phi}$, $\bar{\psi} = \tan(h\partial_x)\,\bar{\phi}$. The velocity potential and the stream function can then be rewritten

$$\phi(x, y, t) = [\cos(y\partial_x) - \sin(y\partial_x)\tan(h\partial_x)]\ \bar{\phi}(x, t) \qquad \text{(A 3a)}$$

$$= \tfrac{1}{2}\,\bar{\phi}(z, t) + \tfrac{1}{2}\,\bar{\phi}(z^*, t) - \tfrac{1}{2\mathrm{i}}\,\bar{\psi}(z, t) + \tfrac{1}{2\mathrm{i}}\,\bar{\psi}(z^*, t), \qquad \text{(A 3b)}$$

$$\psi(x, y, t) = [\sin(y\partial_x) + \cos(y\partial_x)\tan(h\partial_x)]\ \bar{\phi}(x, t) \qquad \text{(A 4a)}$$

$$= \tfrac{1}{2}\,\bar{\psi}(z, t) + \tfrac{1}{2}\,\bar{\psi}(z^*, t) + \tfrac{1}{2\mathrm{i}}\,\bar{\phi}(z, t) - \tfrac{1}{2\mathrm{i}}\,\bar{\phi}(z^*, t). \qquad \text{(A 4b)}$$

Relations (A 3)–(A 4) – together with $\bar{\psi} = \tan(h\partial_x)\bar{\phi}$ – are equivalent to (A 1)–(A 2) but they can deal with infinite depth as well as with shallow water. It is hence sufficient to know $\bar{\phi}$ to determine ϕ and ψ in the whole fluid domain. In the case of infinite depth, $\tan(h\partial_x)$ is the classical Hilbert transform. These relations can be used as renormalization formulae to improve asymptotic deep-water theories (e.g. nonlinear Schrödinger, Benjamin–Ono). One can then expect to find accurate approximations valid for shallow-water from deep-water approximations.

Appendix B. Algebraic versus classical solitary waves

An approximation of the classical solitary wave is obtained from the renormalized cnoidal wave on finite depth (2.3) with $m=1$, i.e.

$$\frac{\kappa\phi}{A} = \frac{\tanh\kappa\theta}{1-\operatorname{sech}^2\kappa\theta\,\sin^2\kappa(y+h)} = \frac{\sinh 2\kappa\theta}{\cosh 2\kappa\theta+\cos 2\kappa(y+h)}. \tag{B 1}$$

Introducing the change of parameters

$$\kappa = \frac{\pi}{2}\frac{h+\ell}{h^2}, \qquad A = \frac{\pi^2}{4}\frac{\ell^2}{h^2}A^\star, \tag{B 2}$$

and taking $h\to\infty$, one obtains

$$\frac{\phi}{\ell A^\star} = \frac{\ell\,\theta}{\theta^2+(y+\ell)^2}, \tag{B 3}$$

which is the velocity potential of the algebraic solitary wave derived in § 9. Thus, in that sense, an algebraic solitary wave is the limit of a classical solitary wave when the depth is infinite.

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